A taste of topos theory

Mark Hopkins

August 2020, **fp-syd**

@antiselfdual
mjhopkins.github.io

Categories

Categories

Objects, arrows, composition, identities.

The category Set

Set

Objects sets

Arrows functions

Initial object

$$0 = \varnothing \xrightarrow{!} X$$

Terminal object

$$X \xrightarrow{!} \{*\} = 1$$

Point

$$
\begin{aligned}
1 &\longrightarrow \{1, 2, 3, 4\} \\
* &\longmapsto 3
\end{aligned}
$$

Sums

$$\{a, b, c\} \uplus \{c, d\} = \{(a, 0), (b, 0), (c, 0), (c, 1), (d, 1)\}$$

Products

$$
\begin{aligned}
\{a, b, c\} \times \{c, d, e, f\} = \{&(a, c), (a, d), (a, e), (a, f), \\
&(b, c), (b, d), (b, e), (b, f), \\
&(c, c), (c, d), (c, e), (c, f)\}
\end{aligned}
$$

Function objects

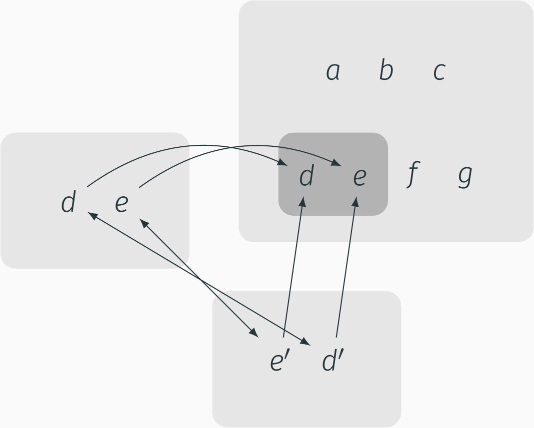
We can form the set $Y^X = \{\text{functions from } X \text{ to } Y\}$.

Subsets

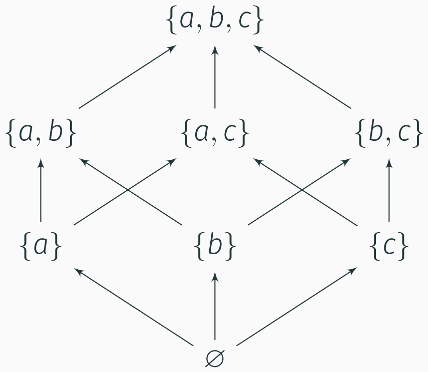
$$\{b, c\} \subseteq \{a, b, c, d\}$$

Almost the same as an injective function.

Subobjects (in Set)



Powersets



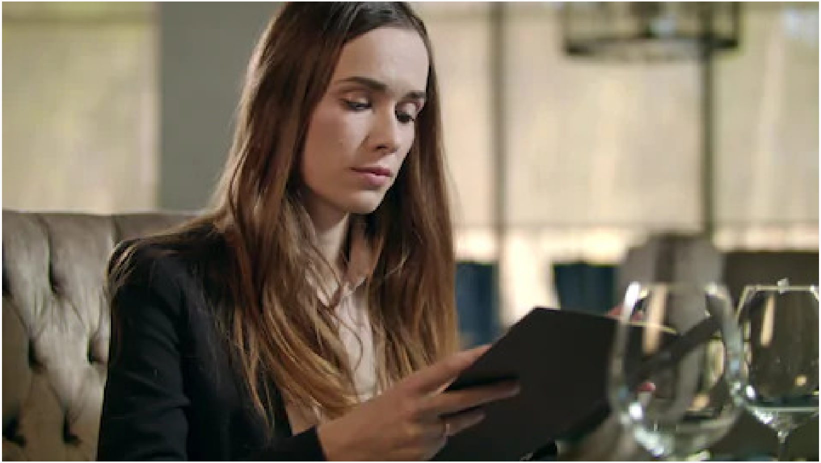
$$\mathcal{P}\{a,b,c\} = \{\varnothing, \{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{a,c\}, \{a,b,c\}\}$$

Characteristic functions

$$\mathsf{Bool} = \{T, F\}$$

$$A \subseteq X \quad \longleftrightarrow \quad \chi_A : X \to \mathsf{Bool}$$

$$A = \{x \in X \mid \chi_A(x)\} = \chi_A^{-1}(T)$$



Logic in Bool $\longleftrightarrow$ "Geometry" in Set

$$
\begin{aligned}
A \cup B &= \{x \mid \chi_A(x) \vee \chi_B(x)\} \\
A \cap B &= \{x \mid \chi_A(x) \wedge \chi_B(x)\} \\
\varnothing &= \{x \mid F\} \\
X &= \{x \mid T\}
\end{aligned}
$$

Functor categories

Functors

A functor between categories is a structure-preserving mapping.

$$
\begin{array}{rccc}
F : & \mathcal{C} & \longrightarrow & \mathcal{D} \\
F : & \mathcal{C}(c, c') & \longrightarrow & \mathcal{D}(Fc, Fc')
\end{array}
$$

$$
\begin{aligned}
F(g\,f) &= (Fg)\,(Ff) \\
F(1_c) &= 1_{Fc}
\end{aligned}
$$

Natural transformations

$$F : \mathcal{C} \to \mathcal{D}$$
$$G : \mathcal{C} \to \mathcal{D}$$

The images of $\mathcal{C}$ under F and G are pictures of $\mathcal{C}$ inside $\mathcal{D}$.

A natural transformation from F to G morphs F's picture of $\mathcal{C}$ into G's.

Natural transformations

A natural transformation between two functors F, G with the same source and target categories is:

for every object c in the source category, an arrow in the target category from Fc to Gc.

But these arrows have to cooperate with the images of arrows under F and G:

$$\begin{array}{ccc} Fc & \xrightarrow{n_c} & Gc \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle Gf} \\ Fc' & \xrightarrow{n_{c'}} & Gc' \end{array}$$

$[\mathcal{C}, \mathcal{D}]$

Functor category $[\mathcal{C}, \mathcal{D}]$

Objects functors from $\mathcal{C}$ to $\mathcal{D}$

Arrows an arrow from F to G is a natural transformation from F to G

Presheaves

A **presheaf** is a functor $C^{op} \to \mathsf{Set}$.

We call $[C^{op}, \mathsf{Set}]$ a presheaf category.

Representables

If $A \in \mathcal{C}$, we can define a presheaf $\underline{A} = \mathcal{C}(-, A)$.

$$
\begin{array}{rcccc}
\underline{A} : C & \longrightarrow & \mathsf{Set} & & \\
B & \longmapsto & C(B, A) & & \\
(f : B \to B') & \longmapsto & \big(\mathcal{C}(B', A) & \xrightarrow{- \circ f} & \mathcal{C}(B, A)\big) \\
 & & g & \longmapsto & g \circ f
\end{array}
$$

[1, Set]

The category 1

[1, Set]

$$[1, \mathsf{Set}] \;\cong\; \mathsf{Set}$$

$$\begin{aligned} F : 1 &\longrightarrow \mathsf{Set} \\ * &\longmapsto F_* \\ 1_* &\longmapsto 1_{F_*} \end{aligned}$$

A natural transformation $F \longrightarrow G$ is just a function $F_* \longrightarrow G_*$.

Representables in [1, Set]

$$\begin{aligned} \underline{1} : 1 &\longrightarrow \mathsf{Set} \\ * &\longmapsto \{1_*\} \end{aligned}$$

Subobjects of $\underline{1}$

Subobjects of $\underline{\mathbf{1}} = \{\underline{\mathbf{0}}, \underline{\mathbf{1}}\}$

$$\underline{\mathbf{0}} : 1 \longrightarrow \mathsf{Set}$$

$$* \longmapsto \varnothing$$

$$\{\underline{\mathbf{0}}, \underline{\mathbf{1}}\} \cong \mathsf{Bool}$$

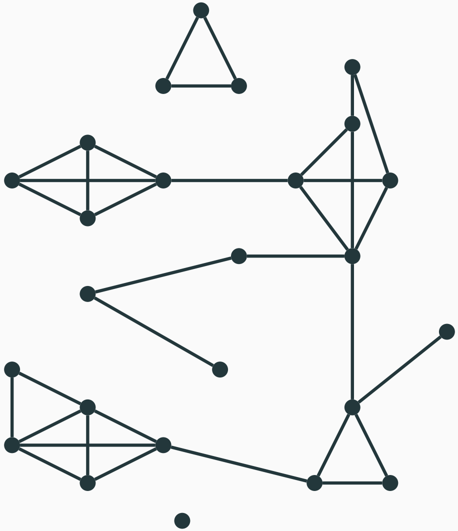
Subfunctors

A **subfunctor** F of G is an equivalence class of monomorphisms $F \to G$.

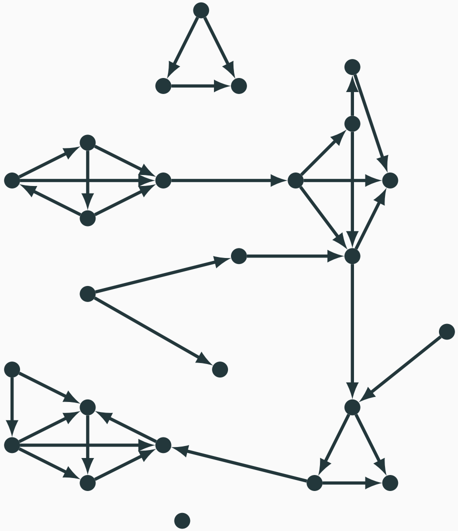
A monomorphism between functors is a natural transformation where every component is an monomorphism.

What is a graph?

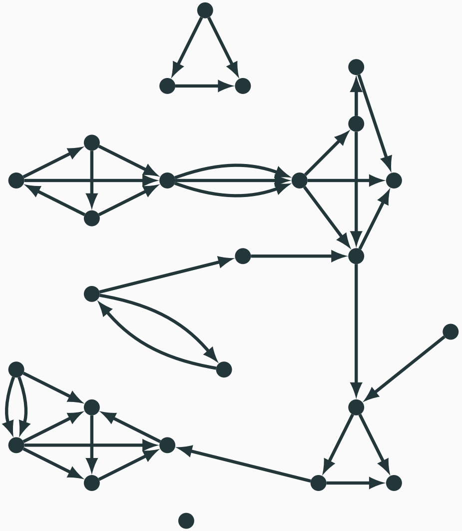
Nodes and edges



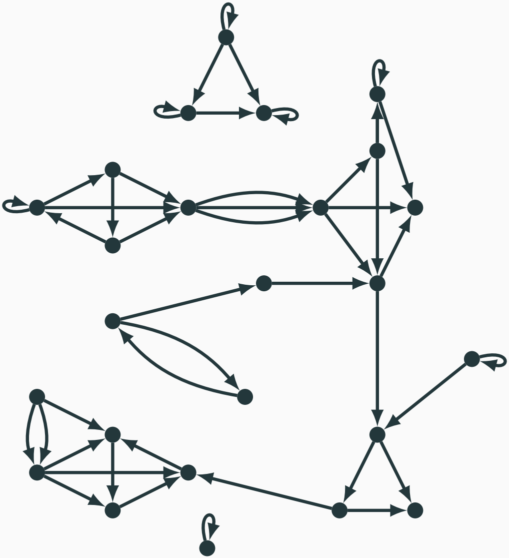
Directed?



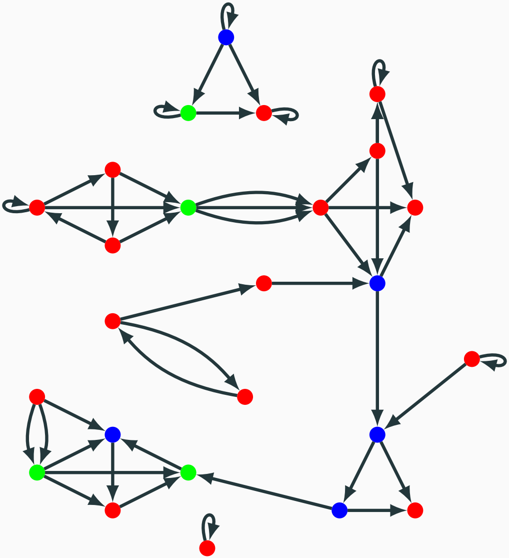
Multiple edges?



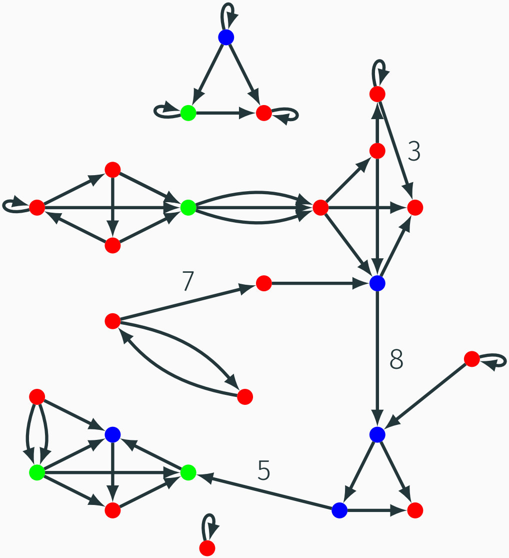
Loops?



Node labels?



Edge labels?



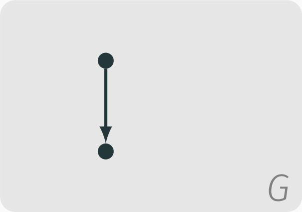
Even crazier

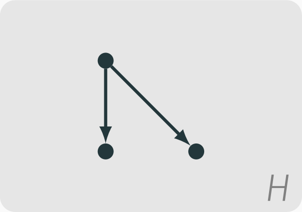
- Hypergraphs
- Simplicial/globular/cubical sets

Map nodes to nodes and edges to edges

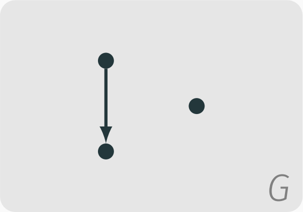
(but the nodes have to follow the edges)

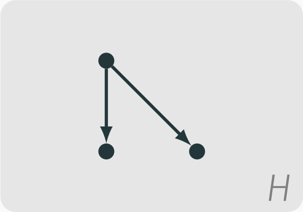
Graph homomorphisms



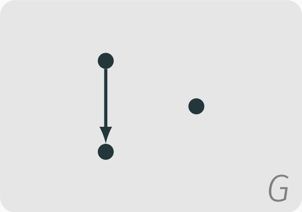


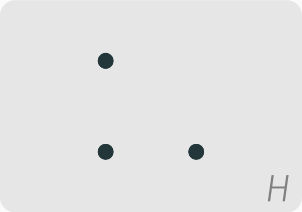
Graph homomorphisms





Graph homomorphisms



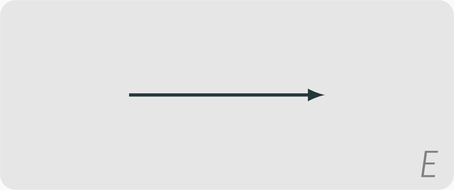


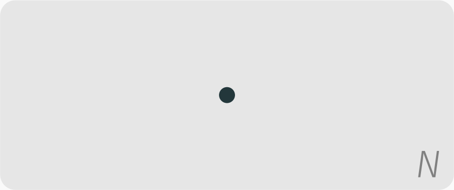
Categories as specifications

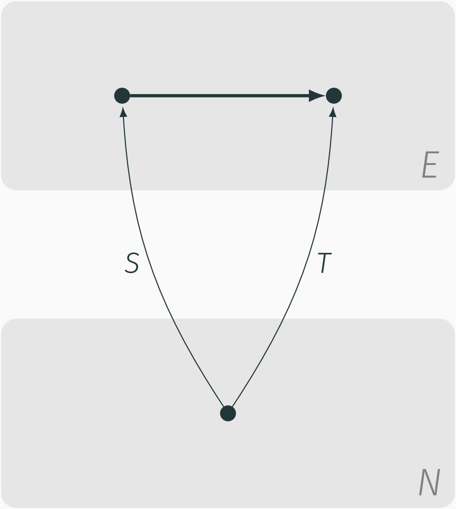
Explain it to me like I’m five

E

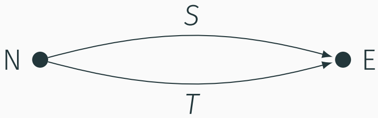
N


E


N


E
S
T
N

So we get a category $\mathcal{G}$



Graphs

The *category of graphs* Graph is the functor category $[\mathcal{G}^{\mathrm{op}}, \mathsf{Set}]$.

A graph X is a functor $\mathcal{G}^{\mathrm{op}} \longrightarrow \mathsf{Set}$.

In other words

- sets X_N, X_E
- functions $s := X_S, t := X_T$

$$s, t : X_E \to X_N$$

A graph homomorphism $n : G \to H$ is a natural transformation $G \to H$ i.e.

$$
\begin{array}{ccc}
G_N & \xrightarrow{n_N} & H_N \\
{\scriptstyle s_G}\uparrow & & \uparrow{\scriptstyle s_H} \\
G_E & \xrightarrow{n_E} & H_E \\
{\scriptstyle t_G}\downarrow & & \downarrow{\scriptstyle t_H} \\
G_N & \xrightarrow{n_N} & H_N
\end{array}
$$

Initial object

The empty graph

$$0_N = \varnothing$$

$$0_E = \varnothing$$

Terminal object

$$1_N = \{*\}$$
$$1_E = \{*\}$$

Points

A point of G is a graph homomorphism $1 \longrightarrow G$.

In other words a *loop* in G.

Sums

$$(G + H)_N = G_N \uplus H_N$$
$$(G + H)_E = G_E \uplus H_E$$

$$s(e) = \begin{cases} s_G(e) \text{ if } e \in G_E \\ s_H(e) \text{ if } e \in H_E \end{cases}$$
$$t(e) = \begin{cases} t_G(e) \text{ if } e \in G_E \\ t_H(e) \text{ if } e \in H_E \end{cases}$$

Products

$$(G \times H)_N = G_N \times H_N$$
$$(G \times H)_E = G_E \times H_E$$

$$s(e1, e2) = (s(e_1), s(e_2))$$
$$t(e1, e2) = (t(e_1), t(e_2))$$

Function objects

We can form the graph H^G of graph homomorphisms from G to H.

$$(H^G)_N = \mathsf{Graph}(N \times G, H)$$
$$(H^G)_E = \mathsf{Graph}(E \times G, H)$$

Representables

There are two representables $\underline{N}$, $\underline{E}$.

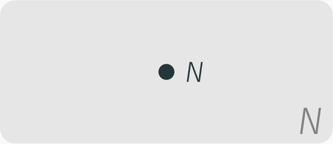
$$\underline{N}_N = \{1_N\} = \{N\} \qquad \underline{E}_N = \{S, T\}$$
$$\underline{N}_E = \varnothing \qquad \underline{E}_E = \{1_E\} = \{E\}$$

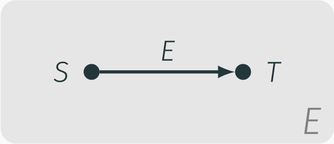
Representables

There are two representables $\underline{N}$, $\underline{E}$.

$$\underline{N}_N = \{1_N\} = \{N\} \qquad \underline{E}_N = \{S, T\}$$

$$\underline{N}_E = \varnothing \qquad \underline{E}_E = \{1_E\} = \{E\}$$





Yoneda lemma

If $\underline{C}$ is a representable and X is a presheaf then

$$X(C) \cong [\mathcal{C}^{\mathrm{op}}, \mathsf{Set}](\underline{C}, X).$$

For any graph X, we can identify

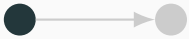
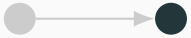
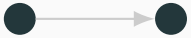
- the set X_N of nodes with the set of graph morphisms $\underline{N} \longrightarrow X$
- the set X_E of edges with the set of graph morphisms $\underline{E} \longrightarrow X$.

Subgraphs of $\underline{N}$

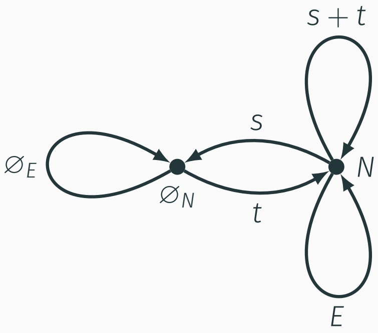
$\underline{N}$ has two subgraphs: 0_N and $\underline{N}$.

Subgraphs of $\underline{E}$

$\underline{E}$ has five subgraphs.

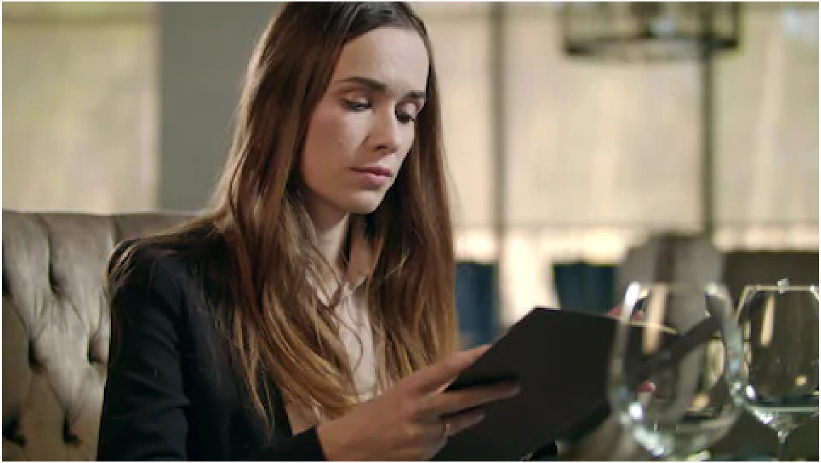
		Source	Target
$\varnothing_E$		$\varnothing_N$	$\varnothing_N$
s		N	$\varnothing_N$
t		$\varnothing_N$	N
$s+t$		N	N
E		N	N

These fit together to make a graph Ω.

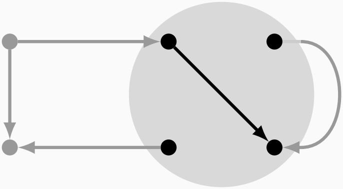


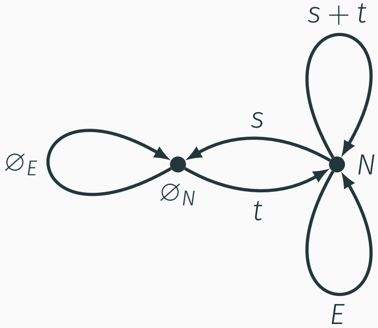
Claim

Ω classifies subgraphs, in the same way that Bool classifies subsets.

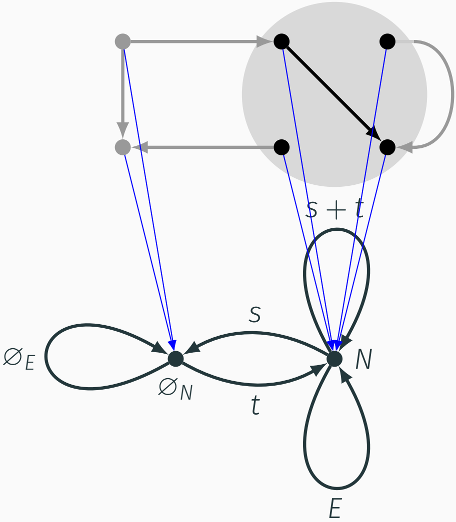


Ω classifies subgraphs!

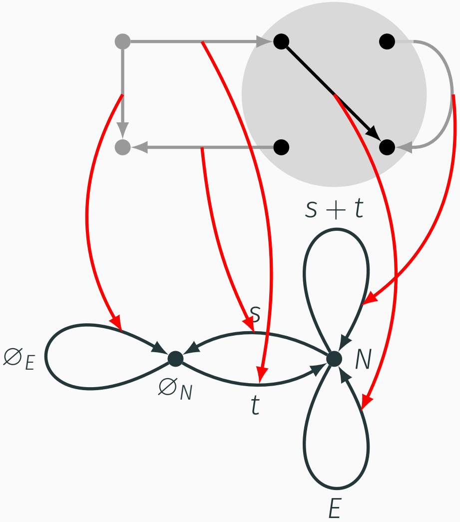




Ω classifies subgraphs!



Ω classifies subgraphs!



Check

Ω has three points.

1 has three subgraphs.

The point $1 \xrightarrow{E} \Omega$ corresponds to "true".

Power graph

We have a *graph* Ω^G of subgraphs of any graph G.

Meet, join, implication

$$\wedge : \Omega \times \Omega \to \Omega$$
$$\vee : \Omega \times \Omega \to \Omega$$
$$\Rightarrow : \Omega \times \Omega \to \Omega$$

Topos

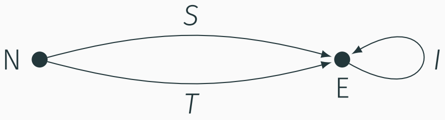
A **topos** is a category that

- is finitely complete (products and equalizers)
- has function objects
- has a subobject classifier

Every presheaf category is a topos!

Other notions of graph

Symmetric graphs

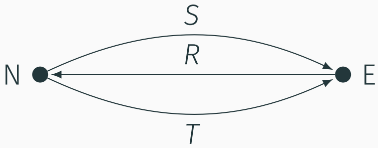


$$IS = T$$

$$IT = S$$

$$I^2 = 1$$

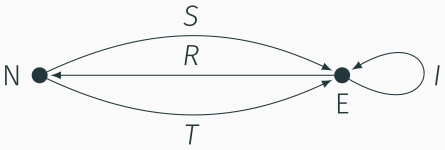
Reflexive graphs



$$SR = 1$$

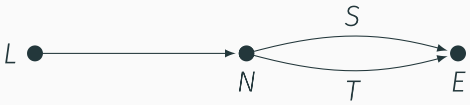
$$TR = 1$$

Reflexive symmetric graphs

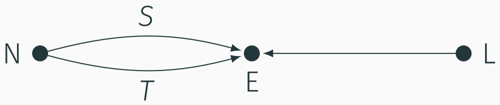


$$IS = T$$
$$IT = S$$
$$I^2 = 1$$
$$SR = 1$$
$$TR = 1$$

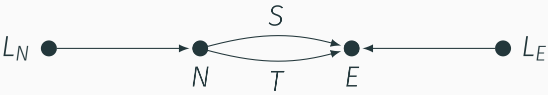
Node-labelled graphs



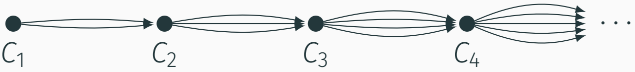
Edge-labelled graphs



Node and Edge-labelled graphs



Semisimplicial sets

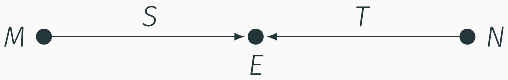


Hypergraphs

Objects $N, E_1, E_2, E_3, \ldots$

Arrows $n_i^n : N \longrightarrow E_n, \quad n = 1, 2, \ldots, \quad i = 1, \ldots, n$

Bipartite graphs



Simple graphs?

Fixing the set of labels?

Quasitopoi

The same, but now Ω only classifies *strong* subobjects.

History

- Leray, 1945
- Cartan seminar, 1948
- Grothendieck, 1955
- Grothendieck, Verdier (SGA), 1962
- Lawvere, Tierney, 1962

Further reading i

- Robert Goldblatt.
  Topoi: the categorial analysis of logic.
  Dover, 2014.
- Peter T. Johnstone.
  Topos Theory.
  Dover Books on Mathematics. Dover Publications, 2014.
- F William Lawvere.
  Qualitative distinctions between some toposes of generalized graphs.
  Categories in computer science and logic (Boulder, CO, 1987), 92:261–299, 1989.

Todd Trimble.
An elementary approach to elementary topos theory.
nLab.

Sebastiano Vigna.
A guided tour in the topos of graphs.
arXiv: Category Theory, 2003.